Algebraic properties of the concurrent star operation

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The theory of traces has two independent origins: combinatorial problems and the theory of concurrent systems. The formal language theory over traces, limited to recognizable and rational trace languages, was developed by Ochmański (see [3]). It is known that rational expressions with classical meaning are useless for expressing recognizable trace languages. For example classical iteration $T^\star$ of a recognizable trace language T need not be recognizable.

Ochmański introduced another meaning of rational expression which he called concurrent meaning. In this way we obtain concurrent iteration $T^\otimes$ of a trace language T which does not put out of the set of recognizable languages. By definition concurrent iteration $T^\otimes$ of T is classical iteration of a language $|T|$ called the decomposition of T. We shall study algebraic properties of those operations.

Let $< M, \cdot, 1 >$ be a monoid and $L \subseteq M$. A monoid morphism $\eta : M \mapsto S$ into a finite monoid $< S, \cdot, 1 >$ *recognizes* L if $\eta^{-1}\eta(L) = L$. The language L is recognizable if there exists a monoid morphism that recognizes L. We denote by *Rec* M the set of all recognizable subests of M.

Let $L, K \subseteq M$. Then $L \cdot K = \{lk \ ; \ l \in L, k \in K\}$ is the product of L and K. By $L^\star$ we denote the (universe of) submonoid of M generated by L. For an alphabet X, $X^\star$ denotes the free monoid generated by X.

The set *Rat* M of all rational languages in M is the smallest set conaining all finite subsets of M and closed on set-theoretical sum $\cup$, product and iteration.

Let X be a finite alphabet and let I be an irreflexive and symmetric relation on X, called *independence relation*. The couple (X, I) is then called a *concurrent alphabet.* The reflexive and symmetric relation $D = X \times X \setminus I$ is called the *dependence relation.* The concurrent alphabet (X, I) induces the set of equations $E = \{ab = ba\ ;\ (a, b) \in I\}$, and the quotient monoid $M(X, I) = X^\star/E$ is called the *trace monoid.* The letter I always denotes independence relation and the letter D denotes dependence relation. Trace monoid will be denoted by $M(X, I)$. Members of trace monoids are called traces and sets of them are called trace languages.

One can extend I and D to $X^\star \times X^\star$: $(u, v) \in I$ iff $alph(u) \times alph(v) \subseteq I$ and $(u, v) \in D$ iff $(alph(u) \times alph(v)) \cap D \neq \emptyset$, and even to $M(X, I) \times M(X, I)$: $(\alpha, \beta) \in I$ iff $\alpha = [u], \beta = [v]$ and $(u, v) \in I$.

Theorem 1 ([3], 6.3.3) *The set Rec M of all recognizable subsets of any trace monoid $M = M(X, I)$ is closed under product: $\forall_{A,B \in Rec\ M} AB \in Rec\ M$.*

Let us recall the definition of connectivity. This quite natural notion is crucial for the theory of recognizable trace languages.

Definition 1 *Let (X, D) be a concurrent alphabet. A word $w \in X^\star$ is connected (with respect to D) iff the graph $(alph(w), (alph(w) \times alph(w)) \cap D)$ is connected; a trace $[w] \in M(X, D)$ is connected iff w is a connected word. The trace language $T \subseteq M(X, D)$ is called connected iff any trace of T is connected.*

Definition 2 *Let $M = M(X, D)$ be a trace monoid and let α, γ be nonempty traces in M. The trace γ is a component of α iff γ is connected and $\alpha = \beta\gamma$ for some $\beta \in M$, such that $alph(\beta) \times alph(\gamma) \subseteq I$ The decomposition of a trace $\alpha \neq [1]$ is the set $|\alpha|$ of all components of α, The decomposition of $[1]$ is defined as $|[1]| = \{[1]\}$. The decomposition of a trace language $T \subseteq M$ is the trace language $|T| = \bigcup\{|\alpha|\ ;\ \alpha \in T\}$.*

Let **Rat** $M =< Rat\ M, \cup, \cdot, ^\star, \emptyset, \{1\} >$ be the algebra of all rational languages in M. The algebra **Rat** M is a homomorphic image of an algebra **Rat** $X^\star$ for suitable X. The set *Rec* M for a trace monoid M is a universe of an algebra of the same similarity type as **Rat** M because the following theorem holds.

Theorem 2 ([3], 6.3.15, 6.3.11) *Let $M = M(X, D)$ be a trace monoid and let $T \subseteq M$ be recognizable. Then languages $|T|, T^\otimes = |T|^\star$ are recognizable.*

In fact *Rec M* is the smallest subset of 2^M containing finite subsets and closed under $\cup, \cdot, |\ |, ^\otimes$. In addition the set *Rat M* is closed under those operations too.
The algebra $\mathbf{Rat}X^\star =< Rat\ X^\star, \cup, \cdot, ^\star, \emptyset, \{1\} >= \mathbf{Reg}\ X$ of all rational (regular) X-languages was studied in literature. One of the main problems concerns its axiomatization (see [1]). It was proved by Redko (1964) that its equational theory is not finitely based. But there exists finite implicational axiom system of the regular sets. The first example of such a system gave Gorshkov and Arkhangelskii (1987). This system is different from that of Kozen.

A *Kleene algebra* (see [2]) is an algebraic structure $\mathrm{K} =< K, +.\cdot, ^\star, 0, 1 >$ satisfying the following:

1. $< K, +.\cdot, 0, 1 >$ is an idempotent semiring with zero and unit;

2. $< K, +.\cdot, ^\star, 0, 1 >$ satisfies the quasiequations:

$$\begin{aligned} 0a &= 0 && (1)\\ a0 &= 0 && (2)\\ 1 + aa^\star &\leq a^\star && (3)\\ 1 + a^\star a &\leq a^\star && (4)\\ b + ax \leq x &\rightarrow a^\star b \leq x && (5)\\ b + xa \leq x &\rightarrow ba^\star \leq x && (6) \end{aligned}$$

where $\leq$ refers to the natural partial order on K: $a \leq b \leftrightarrow a + b = b$.

The natural question arises: Is the algebra $\mathbf{Rec}\ M =< Rec\ M, \cup, \cdot, ^\otimes, \emptyset, \{1\} >$ a Kleene algebra?

EXAMPLE Let $M(X, I)$ be a trace monoid such that $(X, I) = (\{a, b\}, \{(a, b), (b, a)\})$. Let $T = \{[ab]\}$ and $X = \{[w]\ ;\ w \in L(r)\}$, where $L(r)$ is a language defined by regular expression $r = a^+b^+ + 1$. X is recognizable because r is star-connected (Proposition 6.3.11 [3]). Then $T \cdot X \leq X$, but $T^\otimes = |T|^\star = \{[a], [b]\}^\star = M$ and $T^\otimes \cdot X \not\leq X$. So the question has a negative answer.

The situation is completely different in the case of rational languages.

Theorem 3 *For any monoid M the algebra* **Rat** *M is a Kleene algebra.*

Let $M = M(X, I)$ denote a trace monoid.

Lemma 1 *Let $A, T, X \in Rat\ M$ satisfy $A \cup |T|X \subseteq X$, then $T^{\otimes}A \subseteq X$.*

Lemma 1 explains the reason to consider the decomposition operation. Let $\alpha \in M(X, I)$ be a trace. It is obvious that $|\alpha|$ is a finite set of elements which commute with each other. In addition α is a product of all elements of $|\alpha|$ in any order; thus such a decomposition is unique up to a permutation of components.

Lemma 2 *Let X, Y be trace languages of M, then $||X||Y|| \subseteq |X||Y| \cup |X| \cup |Y|$.*

Lemma 3 *Let T be a trace language such that $T \cdot T$ is connected. Then T and $T^{\otimes}$ are connected.*

Corollary 1 *Let T be a trace language, then $|T^{\otimes}| \subseteq T^{\otimes}$ and $T \subseteq T^{\otimes}$.*

We summarize previous results.

Definition 3 *A Generalized Klenee algebra is an algebraic structure $\underline{K} =< K, +, \cdot, \otimes, |\ |, 0, 1 >$ satisfying the following:*

1. *$< K, +. \cdot, 0, 1 >$ is an idempotent semiring with zero and unit with annihilating element 0: $0a = a0 = 0$;*

2. *$< K, +. \cdot, \otimes, |\ |, 0, 1 >$ satisfies the quasiequations:*

$$
\begin{aligned}
|0| &= 0 && (7)\\
|1| &= 1 && (8)\\
|a+b| &= |a| + |b| && (9)\\
||a|| &= |a| && (10)\\
||a||b|| &\leq |a||b| + |a| + |b| && (11)\\
|a^{\otimes}| &\leq a^{\otimes} && (12)\\
a &\leq a^{\otimes} && (13)\\
1 + |a|a^{\otimes} &\leq a^{\otimes} && (14)\\
1 + a^{\otimes}|a| &\leq a^{\otimes} && (15)\\
b + |a|x \leq x &\rightarrow a^{\otimes}b \leq x && (16)\\
b + x|a| \leq x &\rightarrow ba^{\otimes} \leq x && (17)\\
a \leq |b| &\rightarrow a = |a| && (18)\\
|a^2| = a^2 &\rightarrow |a| = a && (19)\\
|a^2| = a^2 &\rightarrow a^{\otimes} = |a^{\otimes}| && (20)
\end{aligned}
$$

where $\leq$ refers to the natural partial order on K: $a \leq b \leftrightarrow a + b = b$.

Lemma 4 *For every trace monoid M the algebra $\mathbf{Rat}_C\ M =< Rat\ M, \cdot, \otimes, |\ |, \emptyset, \{\varepsilon\} >$ is generalized Kleene algebra.*

Theorem 4 *The following quasiequations are theorems of generalized Kleene algebras.*

$$a^{\otimes}a^{\otimes} = a^{\otimes} \quad (21)$$
$$a^{\otimes\otimes} = a^{\otimes} \quad (22)$$
$$1 + aa^{\otimes} \leq a^{\otimes} \quad (23)$$
$$1 + a^{\otimes}a \leq a^{\otimes} \quad (24)$$
$$a \leq b^{\otimes} \rightarrow a^{\otimes} \leq b^{\otimes} \quad (25)$$
$$a \leq b \rightarrow a^{\otimes} \leq b^{\otimes} \quad (26)$$
$$1 + |a|a^{\otimes} = a^{\otimes} \quad (27)$$
$$1 + a^{\otimes}|a| = a^{\otimes} \quad (28)$$

PROBLEM Is the definition of generalized Kleene algebras implicational system of axioms for rational or recognizable trace languages?

References

[1] CONWAY, J. H., *Regular Algebra and Finite Machines*, Chapman and Hall, 1971.

[2] KOZEN, D., *A completeness theorem for Kleene algebras and the algebra of regular events* , Information and Computation 110 (1994), 366-390.

[3] OCHMAŃSKI, E., *Recognizable trace languages*, in *The Book of Traces* ed. V. Dekert, G. Rozenberg, World Scientific 1995, 167-203.